

On the power of a unique quantum witness

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Abstract

In a celebrated paper, Valiant and Vazirani [28] raised the question of whether the difficulty of NP-complete problems was due to the wide variation of the number of witnesses of their instances. They gave a strong negative answer by showing that distinguishing between instances having zero or one witnesses is as hard as recognizing NP, under randomized reductions.

We consider the same question in the quantum setting and investigate the possibility of reducing quantum witnesses in the context of the complexity class QMA, the quantum analogue of NP. The natural way to quantify the number of quantum witnesses is the dimension of the witness subspace W in some appropriate Hilbert space \mathcal{H} . We present an efficient deterministic procedure that reduces any problem where the dimension d of W is bounded by a polynomial to a problem with a unique quantum witness. The main idea of our reduction is to consider the Alternating subspace of the tensor power $\mathcal{H}^{\otimes d}$. Indeed, the intersection of this subspace with $W^{\otimes d}$ is one-dimensional, and therefore can play the role of the unique quantum witness.

1 Introduction

One of the most fundamental ideas of modern complexity theory is that, the study of decision making procedures involving a single party should be extended to the study of more complex procedures where several parties interact. The notions of *verification* and *witness* are at the heart of those complexity classes whose definition inherently involves interaction. The complexity classes P is the set of languages decidable by a polynomial-time deterministic algorithm. Similarly, BPP is the set of promise problems decidable by a polynomial-time bounded-error randomized algorithm. We can think of such an algorithm as a verifier acting alone. The simplest interactive extensions of P and BPP are their non-deterministic analogues, respectively NP and MA [10, 6]. These classes involve also an all powerful prover that sends a single message which is used by the verifier's decision making procedure together with the input. We require that on positive instances there is some message (called in that case a witness) that makes the verifier accept, whereas on negative instances the verifier rejects independently of the message sent by the prover. In the case of MA we can fix the permitted error of the verifier, rather arbitrarily to any constant, say $1/3$.

Quantum complexity classes are often defined by analogy to their classical counterparts. Since quantum computation is inherently probabilistic, the quantum analog of MA is considered to be the right definition of non-deterministic quantum polynomial-time. The quantum extension is twofold: the verifier has the power to decide promise problems in BQP , quantum polynomial-time, and the

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messages he receives from the prover are also quantum. Thus, QMA is the set of promise problems such that on positive instances there exists a quantum witness accepted with probability at least $2/3$ by the polynomial-time quantum verifier and on negative instances the verifier accepts every quantum state with probability at most $1/3$. While the idea that a quantum state might play the role of a witness goes back to Knill [18], the class was formally defined by Kitaev [17] under the name of BQNP. The currently used name QMA was given to the class by Watrous [29]. Kitaev has established several error probability reduction properties of QMA, and proved that the **Local Hamiltonian**, the quantum analog of SAT was complete for it. Watrous has shown that **Group non-Membership** was a problem in QMA and based on this result he has constructed an oracle under which MA is strictly included in QMA. Since then, various problems have been proven to be complete for QMA [14, 16, 21, 15, 22]. A potentially weaker quantum extension of MA, namely QCMA, was defined by Aharonov and Naveh [1]: in the case of QCMA, the verifier is still a quantum polynomial-time algorithm, but the message of the prover can only be classical.

The number of witnesses for positive instances of problems in NP can be exponentially high. Also, known NP-complete problems have different instances with widely varying numbers of solutions. In a celebrated paper, Valiant and Vazirani [28] have raised the question of whether the difficulty of the class NP was due to this wide variation. They gave a strong negative answer to this question in the following sense. Let UP be the set of problems in NP where in addition on positive instances there exists a unique witness. We denote by **PromiseUP** the extention of UP from languages to promise problems. The theorem of Valiant and Vazirani states that any problem in NP can be reduced in randomized polynomial-time to a promise problem in **PromiseUP**, or in set theoretical terms, $NP \subseteq RP^{\text{PromiseUP}}$, where RP is the subclass of problems in BPP where the computation does not err on negative instances. The complexity class UP has also its importance because of its connection to one-way functions: worst case one-way functions exist if and only if $UP \neq P$ [19, 11].

In a recent paper Aharonov, Ben-Or, Brandão and Sattah [3] have asked a similar question for MA, QCMA and QMA. The restriction of the classical-witness classes MA and QCMA to their unique variants UMA and UQCMA is rather natural: no change for negative instances, but on positive instances there has to be exactly one witness that makes the verifier accept with probability at least $2/3$, while all other messages make him accept with probability at most $1/3$. The definition of UQMA, the unique variant of QMA is the following: there is no change for negative instances with respect to QMA, but on positive instances there has to be a quantum witness state $|\psi\rangle$ which is accepted by the verifier with probability at least $2/3$, whereas all states orthogonal to $|\psi\rangle$ are accepted with probability at most $1/3$. Aharonov et al. extended the Valiant–Vazirani proof for the classical witness classes by showing that $MA \subseteq RP^{\text{UMA}}$ and $QCMA \subseteq RP^{\text{UQCMA}}$. On the other hand, they left the existence of a similar result for QMA as an open problem.

Why is it so difficult to reduce the witnesses to a single witness in the quantum case? The basic idea of Valiant and Vazirani is to use pairwise independent universal hash functions, having polynomial size descriptions, that eliminate independently each witness with some constant probability. The size of the original witness set can be guessed approximately by a polynomial-time probabilistic procedure, and in case of a correct guess the hashing keeps alive exactly one witness with again some constant probability. The same idea basically works for MA and QCMA as long as one additional difficulty is overcome: on positive instances there can be exponentially more “pseudo-witnesses”, accepted with probability between $1/3$ and $2/3$, than witnesses which are accepted with probability at least $2/3$. In this case, the Valiant–Vazirani proof technique will eliminate with high probability all witnesses before the elimination of the pseudo-witnesses. The solution of Aharonov et al. for this problem is to divide the interval $(1/3, 2/3)$ into polynomially many smaller intervals and to show that there exists at least one interval such that there are approximately as many witnesses accepted with probability within this interval as above it.

In the quantum case, the set of quantum witnesses can be infinite. For a promise problem in QMA, we can suppose without loss of generality that on positive instances there exists a subspace W such that all unit vectors in W are accepted. The dimension of W could be large and we wish to reduce it to one. Aharonov et. al [3] considered the special case where the dimension of W is two. Although classically two witnesses are trivially reducible to the unique witness case, they have shown that the natural generalization of the Valiant–Vazirani construction cannot solve even the two-dimensional quantum witness case.

Indeed, the natural generalization of the Valiant–Vazirani construction to this situation is to use random projections and hope that some one-dimensional subspace of W will be accepted with substantially higher probability than its orthogonal. A first difficulty is to implement such projections efficiently. But more importantly, a random projection would not create a polynomial gap in the acceptance probabilities for the pure states of W : in fact all states in W which were accepted with exponentially close probabilities, will still be accepted after the random projection with exponentially close probabilities.

Here we describe a fundamentally different proof technique to tackle this problem, which is sufficiently powerful to solve the case when the dimension of the witness subspace W is polynomially bounded in the length of the input. This leads us naturally to the quantum analog of the promise problem class **FewP**. This complexity class was defined by Allender [4] as the set of problems in **NP** with the additional constraint that there is a polynomial q such that on every positive instance of length n , the number of witnesses is at most $q(n)$. The class **FewP** was extensively studied in the context of counting complexity classes [5, 27, 20, 12, 26]. We define **FewQMA**, the quantum analog of **FewP**, as the set of promise problems in **QMA** for which there exists a polynomial q with the following properties: on negative instances every message of the prover is accepted by the verifier with probability at most $1/3$; on a positive instance x there exists a subspace W_x of dimension between 1 and $q(|x|)$, such that all pure states in W_x are accepted with probability at least $2/3$, while all pure states orthogonal to W_x are accepted with probability at most $1/3$. Our main theorem extends the result of Valiant and Vazirani to this complexity class. More precisely, we show that **FewQMA** is deterministic polynomial-time Turing-reducible to **UQMA**.

Main Theorem $\text{FewQMA} \subseteq \text{P}^{\text{UQMA}}$.

The first idea to establish this result is that instead of manipulating the states within the original space \mathcal{H} of dimension K , we consider its t -fold tensor powers $\mathcal{H}^{\otimes t}$. At first glance, this does not seem to be going in the right direction because the dimension of $W^{\otimes t}$ grows as d^t , where d is the dimension of the witness space W . Our second idea is to consider the alternating subspace Alt of $\mathcal{H}^{\otimes t}$ whose dimension is $\binom{K}{t}$. The important thing to notice is that the dimension of the intersection $\text{Alt} \cap W^{\otimes t}$ is equal to one when $t = d$. The reason is that this intersection is in fact equal to the alternating subspace of $W^{\otimes t}$ whose dimension is $\binom{d}{t}$. Therefore, we will choose this one-dimensional subspace as our unique quantum witness. Of course, we don't know exactly the dimension of W , but since we have a polynomial upper bound $q(|x|)$ on it, we just try every possible value t between 1 and $q(|x|)$.

For a fixed t , we would ideally implement $\Pi_{W^{\otimes t}} \cdot \Pi_{\text{Alt}}$, the product of the projection to Alt followed by the projection to $W^{\otimes t}$. The reason that this would work is the following. The unique pure state in $\text{Alt} \cap W^{\otimes t}$ (up to a global phase) is clearly accepted with probability 1. On the other hand, we claim that any state $|\phi\rangle$ orthogonal to that is rejected with probability 1. Indeed, $|\phi\rangle$ can be decomposed as $|\phi_1\rangle + |\phi_2\rangle$, where $|\phi_1\rangle \in \text{Alt}^\perp$ and $|\phi_2\rangle \in W^{\otimes t \perp}$. Therefore $|\phi_1\rangle$ is rejected by Π_{Alt} and $|\phi_2\rangle$ is rejected by $\Pi_{W^{\otimes t}}$. This implies the claim since we can show that the two projectors actually commute.

We can efficiently implement Π_{Alt} by a procedure we call the **Alternating Test**. A similar procedure to ours, implementing efficiently the projection to the symmetric subspace **Sym** of $\mathcal{H}^{\otimes t}$, was proposed by Barenco et al. [7] as the basis of a method for the stabilization of quantum computations. In fact,

in the two-fold tensor product case, the two procedures coincide and become the well known Swap Test which was used by Buhrman et al. [9] for deciding if two given pure states are close or far apart.

We can't implement $\Pi_{W^{\otimes t}}$ exactly, but we can approximate it efficiently by a procedure called the **Witness Test**. This test just applies independently to all the t components of the state the procedure at our disposal which decides in \mathcal{H} whether a state is a witness or not, and accepts if all applications accept. There is only one difficulty left: since Π_{Alt} and the **Witness Test** don't necessarily commute, our previous argument which showed that states in $W^{\otimes t^\perp}$ were rejected with probability 1 doesn't work anymore. We overcome this difficulty by showing that the commutativity of the two projections implies that the projections to **Alt** of such states are also in $W^{\otimes t^\perp}$, and therefore get rejected with high probability by the **Witness Test**.

An interesting feature of our reduction is that it is deterministic, while the Valiant-Vazirani procedure is probabilistic. It is fair to say though that classically the witnesses can all be enumerated when their number is bounded by a polynomial. Therefore, in that case, the reduction can also be done deterministically, implying that $\text{FewP} \subseteq \text{P}^{\text{PromiseUP}}$. We believe that reducing QMA to a unique witness, which this paper leaves as an open question, will require a probabilistic or a quantum procedure.

The rest of the paper is structured as follows. In Section 2 we state some facts about the interaction of the tensor products of subspaces with the alternating subspace. In Section 3 we define the complexity classes we are concerned with. We give two definitions for FewQMA and show that they are equivalent. Section 4 is entirely devoted to the proof of our main result. Finally in the Appendix A we consider a third definition and show a weak equivalence with the previous ones.

2 Preliminaries

In this section we present definitions and lemmas that we will need in the proof of our main result.

We represent by $[t]$ the set $\{1, 2, \dots, t\}$. For a Hilbert space \mathcal{H} , we denote by $\dim(\mathcal{H})$ the dimension of \mathcal{H} . For a subspace S of \mathcal{H} , let S^\perp represent the subspace of \mathcal{H} orthogonal to S , and let Π_S denote the projector onto S . For subspaces S_1, S_2 of \mathcal{H} , their *direct sum* $S_1 + S_2$ is defined as $\text{span}(S_1 \cup S_2)$, and when S_1, S_2 are orthogonal subspaces, we denote their (orthogonal) direct sum by $S_1 \oplus S_2$. The following relations are standard.

Fact 1 1. Let S_1, S_2 be subspaces of a Hilbert space \mathcal{H} . Then $(S_1 \cap S_2)^\perp = S_1^\perp + S_2^\perp$.

2. Let S_1, S_2 be subspaces of Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$ respectively. Then, $(S_1 \otimes S_2)^\perp = (S_1^\perp \otimes \mathcal{H}_2) + (\mathcal{H}_1 \otimes S_2^\perp) = (S_1^\perp \otimes \mathcal{H}_2) \oplus (S_1 \otimes S_2^\perp)$.

Let \mathcal{B} represent the two-dimensional complex Hilbert space and let $\{|0\rangle, |1\rangle\}$ be the computational basis for \mathcal{B} . For a natural number k , the computational basis of $\mathcal{B}^{\otimes k}$ (the k -fold tensor of \mathcal{B}) consists of $\{|r\rangle : r \in \{0, 1\}^k\}$, where $|r\rangle$ denotes the tensor product $|r_1\rangle \otimes \dots \otimes |r_k\rangle$ for the k -bit string $r = r_1 \dots r_k$. Fix k and let \mathcal{H} denote $\mathcal{B}^{\otimes k}$ and let $K = 2^k$. By a *pure state* in \mathcal{H} , we mean a unit vector in \mathcal{H} . A *mixed state* or just *state* is a positive semi-definite operator in \mathcal{H} with trace 1. We refer the reader to the text [24] for concepts related to quantum information theory. For a natural number $t \in [K]$, we will think of states of $\mathcal{H}^{\otimes t}$ as consisting of t registers, where the content of each register is a state with support in \mathcal{H} .

We will consider the interaction of $W^{\otimes t}$, where W is a d -dimensional subspace of \mathcal{H} for some d satisfying $2 \leq t \leq d \leq K$, with the alternating and symmetric subspaces of $\mathcal{H}^{\otimes t}$. Let S_t denote the set of all permutations $\pi : [t] \rightarrow [t]$. For a permutation $\pi \in S_t$, let the unitary operator U_π , acting on $\mathcal{H}^{\otimes t}$, be given by $|s_{\pi(1)}\rangle \otimes \dots \otimes |s_{\pi(t)}\rangle$.

For permutations π_1, π_2 , let $\pi_1 \circ \pi_2$ represent their composition. It is easily seen that $U_{\pi_1 \circ \pi_2} = U_{\pi_1} U_{\pi_2}$. For distinct $i, j \in [t]$, let π_{ij} be the transposition of i and j . For all distinct $i, j \in [t]$, the

symmetric subspace of $W^{\otimes t}$ with respect to i and j is given by

$\text{Sym}_{ij}^{W^{\otimes t}} = \{|\phi\rangle \in W^{\otimes t} : U_{\pi_{ij}}|\phi\rangle = |\phi\rangle\}$, and the *symmetric subspace* of $W^{\otimes t}$ is defined as $\text{Sym}^{W^{\otimes t}} = \bigcap_{i \neq j} \text{Sym}_{ij}^{W^{\otimes t}}$. Similarly, for all distinct $i, j \in [t]$, the alternating subspace of $W^{\otimes t}$ with respect to i and j is defined as $\text{Alt}_{ij}^{W^{\otimes t}} = \{|\phi\rangle \in W^{\otimes t} : U_{\pi_{ij}}|\phi\rangle = -|\phi\rangle\}$, and the *alternating subspace* of $W^{\otimes t}$ is defined as $\text{Alt}^{W^{\otimes t}} = \bigcap_{i \neq j} \text{Alt}_{ij}^{W^{\otimes t}}$.

The subspaces $\text{Sym}^{W^{\otimes t}}$ and $\text{Alt}^{W^{\otimes t}}$ are of dimension $\binom{d+t-1}{t}$ and $\binom{d}{t}$ respectively [8]. In particular, $\text{Alt}^{\mathcal{H}^{\otimes 2}}$ and $\text{Sym}^{\mathcal{H}^{\otimes 2}}$ have respective dimensions $\binom{K+1}{2}$ and $\binom{K}{2}$ and since they are orthogonal, we have $\mathcal{H}^{\otimes 2} = \text{Alt}^{\mathcal{H}^{\otimes 2}} \oplus \text{Sym}^{\mathcal{H}^{\otimes 2}}$. This implies that for every distinct $i, j \in [t]$, we have $\text{Alt}_{ij}^{W^{\otimes t}} \oplus \text{Sym}_{ij}^{W^{\otimes t}} = W^{\otimes t}$. It follows that

Claim 1 $(\text{Alt}^{W^{\otimes t}})^\perp \cap W^{\otimes t} = \sum_{i \neq j} \text{Sym}_{ij}^{W^{\otimes t}}$.

Proof Since $\text{Alt}_{ij}^{W^{\otimes t}} \oplus \text{Sym}_{ij}^{W^{\otimes t}} = W^{\otimes t}$, we have $(\text{Alt}_{ij}^{W^{\otimes t}})^\perp = \text{Sym}_{ij}^{W^{\otimes t}} \oplus (W^{\otimes t})^\perp$. Therefore

$$\begin{aligned} (\text{Alt}^{W^{\otimes t}})^\perp \cap W^{\otimes t} &= ((\bigcap_{i \neq j} \text{Alt}_{ij}^{W^{\otimes t}})^\perp) \cap W^{\otimes t} \quad (\text{from definition of Alt}^{W^{\otimes t}}) \\ &= (\sum_{i \neq j} (\text{Alt}_{ij}^{W^{\otimes t}})^\perp) \cap W^{\otimes t} \quad (\text{from Fact 1}) \\ &= (\sum_{i \neq j} (\text{Sym}_{ij}^{W^{\otimes t}} \oplus (W^{\otimes t})^\perp)) \cap W^{\otimes t} \\ &= ((\sum_{i \neq j} \text{Sym}_{ij}^{W^{\otimes t}}) \oplus (W^{\otimes t})^\perp) \cap W^{\otimes t} = \sum_{i \neq j} \text{Sym}_{ij}^{W^{\otimes t}}. \end{aligned}$$

The last equality holds since $(\sum_{i \neq j} \text{Sym}_{ij}^{W^{\otimes t}}) \subseteq W^{\otimes t}$. \square

Note that for $W = \mathcal{H}$ the claim states that $(\text{Alt}^{\mathcal{H}^{\otimes t}})^\perp = \sum_{i \neq j} \text{Sym}_{ij}^{\mathcal{H}^{\otimes t}}$. For us, a particularly important case is when the number of registers t is equal to d , the dimension of the subspace W . Then the alternating subspace $\text{Alt}^{W^{\otimes d}}$ is one-dimensional. Let $\{|\psi_1\rangle, \dots, |\psi_d\rangle\}$ be any orthonormal basis of W , and let the vector $|W_{alt}\rangle \in W^{\otimes d}$ be defined as $|W_{alt}\rangle = \frac{1}{\sqrt{d!}} \sum_{\pi \in S_d} \text{sgn}(\pi) U_\pi |\psi_1\rangle \dots |\psi_d\rangle$, where $\text{sgn}(\pi)$ denotes the sign of the permutation π . The following claim states that $|W_{alt}\rangle$ spans the one-dimensional subspace $\text{Alt}^{W^{\otimes d}}$. This immediately implies that $|W_{alt}\rangle$ is independent of the choice of the basis (up to a global phase).

Claim 2 $\text{Alt}^{W^{\otimes d}} = \text{span}\{|W_{alt}\rangle\}$.

Proof We show that $|W_{alt}\rangle \in \text{Alt}^{W^{\otimes d}}$. This implies the statement since $\dim(\text{Alt}^{W^{\otimes d}}) = \binom{d}{d} = 1$. For any distinct $i, j \in [d]$ we show $|W_{alt}\rangle \in \text{Alt}_{ij}^{W^{\otimes d}}$. For a permutation $\pi \in S_d$, we set $\pi' = \pi_{ij} \circ \pi$. We then have

$$\begin{aligned} &U_{\pi_{ij}}|W_{alt}\rangle \\ &= U_{\pi_{ij}} \frac{1}{\sqrt{d!}} \sum_{\pi \in S_d} \text{sgn}(\pi) U_\pi |\psi_1\rangle \otimes \dots \otimes |\psi_d\rangle = \frac{1}{\sqrt{d!}} \sum_{\pi \in S_d} \text{sgn}(\pi) U_{\pi_{ij} \circ \pi} |\psi_1\rangle \otimes \dots \otimes |\psi_d\rangle \\ &= \frac{1}{\sqrt{d!}} \sum_{\pi' \in S_d} \text{sgn}(\pi_{ij}^{-1} \circ \pi') U_{\pi'} |\psi_1\rangle \otimes \dots \otimes |\psi_d\rangle = (-1) \cdot \frac{1}{\sqrt{d!}} \sum_{\pi' \in S_d} \text{sgn}(\pi') U_{\pi'} |\psi_1\rangle \otimes \dots \otimes |\psi_d\rangle \\ &= -|W_{alt}\rangle, \end{aligned}$$

where we used that $\text{sgn}(\pi_{ij}^{-1} \circ \pi') = -\text{sgn}(\pi')$. \square

Next, we show that the projections on the spaces $\text{Alt}^{\mathcal{H}^{\otimes t}}$ and $W^{\otimes t}$ commute for any $2 \leq t \leq d$.

Claim 3 Let $2 \leq t \leq d$. Then, $\Pi_{\text{Alt}^{\mathcal{H}^{\otimes t}}} \cdot \Pi_{W^{\otimes t}} = \Pi_{W^{\otimes t}} \cdot \Pi_{\text{Alt}^{\mathcal{H}^{\otimes t}}}$.

Proof Set $T = (\text{Alt}^{W^{\otimes t}})^\perp \cap W^{\otimes t}$. Then $W^{\otimes t} = \text{Alt}^{W^{\otimes t}} \oplus T$ and hence, $\Pi_{W^{\otimes t}} = \Pi_{\text{Alt}^{W^{\otimes t}}} + \Pi_T$. We have

$$\begin{aligned} T &= (\text{Alt}^{W^{\otimes t}})^\perp \cap W^{\otimes t} \\ &= \sum_{i \neq j} \text{Sym}_{ij}^{W^{\otimes t}} \quad (\text{from Claim 1}) \\ &\subseteq \sum_{i \neq j} \text{Sym}_{ij}^{\mathcal{H}^{\otimes t}} \quad (\text{by definition}) \\ &= (\text{Alt}^{\mathcal{H}^{\otimes t}})^\perp \quad (\text{from Claim 1}) . \end{aligned}$$

This implies that $\Pi_{\text{Alt}^{\mathcal{H}^{\otimes t}}} \cdot \Pi_T = \Pi_T \cdot \Pi_{\text{Alt}^{\mathcal{H}^{\otimes t}}} = \mathbf{0}$. Also, $\text{Alt}^{W^{\otimes t}} \subseteq \text{Alt}^{\mathcal{H}^{\otimes t}}$ since $\text{Alt}^{W^{\otimes t}} = \text{Alt}^{\mathcal{H}^{\otimes t}} \cap W^{\otimes t}$. Therefore, we have

$$\Pi_{\text{Alt}^{\mathcal{H}^{\otimes t}}} \cdot \Pi_{W^{\otimes t}} = \Pi_{\text{Alt}^{\mathcal{H}^{\otimes t}}} \cdot (\Pi_{\text{Alt}^{W^{\otimes t}}} + \Pi_T) = \Pi_{\text{Alt}^{W^{\otimes t}}} .$$

Similarly

$$\Pi_{W^{\otimes t}} \cdot \Pi_{\text{Alt}^{\mathcal{H}^{\otimes t}}} = \Pi_{\text{Alt}^{W^{\otimes t}}} .$$

Hence $\Pi_{\text{Alt}^{\mathcal{H}^{\otimes t}}} \cdot \Pi_{W^{\otimes t}} = \Pi_{W^{\otimes t}} \cdot \Pi_{\text{Alt}^{\mathcal{H}^{\otimes t}}}$. \square

This commutativity relation enables us to derive the following property

Claim 4 For any state $|\phi\rangle \in (\text{Alt}^{W^{\otimes d}})^\perp$, we have $\Pi_{\text{Alt}^{\mathcal{H}^{\otimes d}}} |\phi\rangle \in (W^{\otimes d})^\perp$.

Proof First note that $(\text{Alt}^{W^{\otimes d}})^\perp = (\text{Alt}^{\mathcal{H}^{\otimes d}} \cap W^{\otimes d})^\perp$ and by Fact 1, $(\text{Alt}^{W^{\otimes d}})^\perp = (\text{Alt}^{\mathcal{H}^{\otimes d}})^\perp + (W^{\otimes d})^\perp$. Hence we can decompose $|\phi\rangle$ as $|\phi_1\rangle + |\phi_2\rangle$, where $|\phi_1\rangle \in (\text{Alt}^{\mathcal{H}^{\otimes d}})^\perp$ and $|\phi_2\rangle \in (W^{\otimes d})^\perp$. As $\Pi_{\text{Alt}^{\mathcal{H}^{\otimes d}}} |\phi_1\rangle = \mathbf{0}$, it suffices to show that $\Pi_{\text{Alt}^{\mathcal{H}^{\otimes d}}} |\phi_2\rangle \in (W^{\otimes d})^\perp$. For this, we prove that $\Pi_{(W^{\otimes d})^\perp} \cdot \Pi_{\text{Alt}^{\mathcal{H}^{\otimes d}}} |\phi_2\rangle = \Pi_{\text{Alt}^{\mathcal{H}^{\otimes d}}} |\phi_2\rangle$.

Claim 3 implies that

$$\Pi_{\text{Alt}^{\mathcal{H}^{\otimes d}}} \cdot \Pi_{(W^{\otimes d})^\perp} = \Pi_{(W^{\otimes d})^\perp} \cdot \Pi_{\text{Alt}^{\mathcal{H}^{\otimes d}}} .$$

Also, $|\phi_2\rangle = \Pi_{(W^{\otimes d})^\perp} |\phi_2\rangle$ since $|\phi_2\rangle \in (W^{\otimes d})^\perp$. Therefore we can conclude by the following equalities:

$$\begin{aligned} \Pi_{(W^{\otimes d})^\perp} \cdot \Pi_{\text{Alt}^{\mathcal{H}^{\otimes d}}} |\phi_2\rangle &= \Pi_{\text{Alt}^{\mathcal{H}^{\otimes d}}} \cdot \Pi_{(W^{\otimes d})^\perp} |\phi_2\rangle \\ &= \Pi_{\text{Alt}^{\mathcal{H}^{\otimes d}}} |\phi_2\rangle . \end{aligned}$$

\square

3 Complexity classes

In this section we define the relevant complexity classes and state the facts needed about them. For a quantum circuit V , we let V also represent the unitary transformation corresponding to the circuit. We call a *verification procedure* a family of quantum circuits $\{V_x : x \in \{0,1\}^*\}$ uniformly generated in polynomial-time, together with polynomials k and m such that V_x acts on $k(|x|) + m(|x|)$ qubits. We refer to the first $k(|x|)$ qubits as *witness qubits* and to the last $m(|x|)$ qubits as *auxiliary qubits*. To simplify notation, when the input x is implicit in the discussion, we refer to $k(|x|)$ by k , and to $m(|x|)$ by m . We will make repeated use of the following projections in $\mathcal{B}^{\otimes(k+m)}$:

$$\Pi_{acc} = |1\rangle\langle 1| \otimes I_{k+m-1}, \quad \Pi_{init} = I_k \otimes |0^m\rangle\langle 0^m|,$$

where I_n is the identity operator on n qubits. We will also make use of the operator Π_x defined as $\Pi_x = \Pi_{init} V_x^\dagger \Pi_{acc} V_x \Pi_{init}$. It is easy to see that Π_x is positive semi-definite.

Given a verification procedure, on input x , a Quantum Merlin-Arthur protocol proceeds in the following way: the prover Merlin sends a pure state $|\psi\rangle \in \mathcal{B}^{\otimes k}$, the *witness*, to the verifier Arthur, who then applies the circuit V_x to $|\psi\rangle \otimes |0^m\rangle$, and accepts if the measurement of the first qubit of the result gives 1. We will denote the probability that Arthur accepts x with witness $|\psi\rangle$ by $\Pr[V_x \text{ outputs Accept on } |\psi\rangle]$, which is equal to $\|\Pi_{acc} V_x (|\psi\rangle \otimes |0^m\rangle)\|^2$.

A promise problem is a tuple $L = (L_{yes}, L_{no})$ with $L_{yes} \cup L_{no} \subseteq \{0, 1\}^*$ and $L_{yes} \cap L_{no} = \emptyset$. We now define the following complexity classes.

Definition 1 *A promise problem $L = (L_{yes}, L_{no})$ is in the complexity class Quantum Merlin-Arthur QMA if there exists a verification procedure $\{V_x : x \in \{0, 1\}^*\}$ with polynomials k and m such that*

1. *for all $x \in L_{yes}$, there exists a witness $|\psi\rangle$, such that $\|\Pi_{acc} V_x (|\psi\rangle \otimes |0^m\rangle)\|^2 \geq 2/3$,*
2. *for all $x \in L_{no}$, and for all witnesses $|\psi\rangle$, $\|\Pi_{acc} V_x (|\psi\rangle \otimes |0^m\rangle)\|^2 \leq 1/3$.*

For a promise problem in QMA, we can suppose without loss of generality that on positive instances there exists a subspace W such that all unit vectors in W are accepted. Hence, by putting a polynomial upper bound on the dimension of this subspace, we derive the following definition:

Definition 2 *Let $c, w, s : \mathbb{N} \rightarrow [0, 1]$ be polynomial-time computable functions such that $c(n) > \max\{w(n), s(n)\}$ for all $n \in \mathbb{N}$. A promise problem $L = (L_{yes}, L_{no})$ is in the complexity class Few Quantum Merlin-Arthur FewQMA(c, w, s) if there exists a verification procedure $\{V_x : x \in \{0, 1\}^*\}$ with polynomials k and m , and a polynomial q such that*

1. *for all $x \in L_{yes}$ there exists a subspace W_x of $\mathcal{B}^{\otimes k}$ with $\dim(W_x) \in [q(|x|)]$ such that*
 - (a) *for all witnesses $|\psi\rangle \in W_x$, $\|\Pi_{acc} V_x (|\psi\rangle \otimes |0^m\rangle)\|^2 \geq c(|x|)$, and*
 - (b) *for all witnesses $|\phi\rangle \in W_x^\perp$, $\|\Pi_{acc} V_x (|\phi\rangle \otimes |0^m\rangle)\|^2 \leq w(|x|)$,*
2. *for all $x \in L_{no}$ and for all pure states $|\psi\rangle \in \mathcal{B}^{\otimes k}$, $\|\Pi_{acc} V_x (|\psi\rangle \otimes |0^m\rangle)\|^2 \leq s(|x|)$.*

Definition 3 *The class Few Quantum Merlin-Arthur FewQMA is FewQMA($2/3, 1/3, 1/3$).*

We next provide an alternative definition of FewQMA(c, w, s) and show that the two definitions are equivalent.

Definition 4 *Let $c, w, s : \mathbb{N} \rightarrow [0, 1]$ be polynomial-time computable functions such that $c(n) > \max\{w(n), s(n)\}$ for all $n \in \mathbb{N}$. A promise problem $L = (L_{yes}, L_{no})$ is in the complexity class Alternative-FewQMA(c, w, s) if there exists a verification procedure $\{V_x : x \in \{0, 1\}^*\}$ with polynomials k and m , and a polynomial q such that*

1. *for all $x \in L_{yes}$, the number of eigenvalues of Π_x which are at least $c(|x|)$ is in $[q(|x|)]$, and no eigenvalue of Π_x is in the open interval $(w(|x|), c(|x|))$.*
2. *for all $x \in L_{no}$, all eigenvalues of Π_x are at most $s(|x|)$.*

We prove the following equivalence between the two definitions.

Theorem 1 *Let $c, w, s : \mathbb{N} \rightarrow [0, 1]$ be polynomial-time computable functions such that $c(n) > \max\{w(n), s(n)\}$ for all $n \in \mathbb{N}$. Then FewQMA(c, w, s) = Alternative-FewQMA(c, w, s).*

Proof Part 1 (Definition 4 \Rightarrow Definition 2): Let L be a promise problem $\text{Alternative-FewQMA}(c, w, s)$ with some verification procedure $\{V_x\}$ and polynomial q according to Definition 4. We show that $\{V_x\}$ and q satisfy also Definition 2 with the same parameters.

We first consider the case $x \in L$. For every $|u\rangle \in \mathcal{B}^{\otimes(k+m)}$, we have

$$\|\Pi_{acc}V_x\Pi_{init}|u\rangle\|^2 = \langle u|\Pi_x|u\rangle . \quad (1)$$

It is easy to check that all eigenvectors of Π_x are also eigenvectors of Π_{init} . The 1-eigenvectors of the projector Π_{init} are of the form $|u\rangle \otimes |0^m\rangle$ with $|u\rangle \in \mathcal{B}^{\otimes k}$, and any vector orthogonal to these is an eigenvector with eigenvalue 0. Let r be the number of eigenvalues of Π_x that are at least $c(|x|)$, by hypothesis $r \in [q(|x|)]$. Let $\{|v_i\rangle \otimes |0^m\rangle : i \in [r]\}$ be a set of orthonormal eigenvectors of Π_x with respective eigenvalues $\{\lambda_i \geq c(|x|) : i \in [r]\}$, we set $W_x = \text{span}(\{|v_i\rangle : i \in [r]\})$. Then all remaining $2^{k+m} - r$ eigenvalues of Π_x are less than or equal to $w(|x|)$. Let $\{|v_i\rangle \otimes |0^m\rangle : i \in \{r+1, \dots, 2^{k+m}\}\}$ be a set of orthonormal eigenvectors with such eigenvalues. It is clear then that $W_x^\perp = \text{span}(\{|v_i\rangle : r < i \leq 2^{k+m}\})$.

We consider a pure state $|\psi\rangle = \sum_{i \in [r]} \alpha_i |v_i\rangle$ in W_x . Then,

$$\begin{aligned} \|\Pi_{acc}V_x(|\psi\rangle \otimes |0^m\rangle)\|^2 &= \|\Pi_{acc}V_x\Pi_{init}(|\psi\rangle \otimes |0^m\rangle)\|^2 \\ &= (\langle \psi | \otimes \langle 0^m |)(\Pi_x|\psi\rangle \otimes |0^m\rangle) \\ &= (\langle \psi | \otimes \langle 0^m |)((\sum_{i \in [r]} \lambda_i \alpha_i \cdot |v_i\rangle) \otimes |0^m\rangle) \\ &= \sum_{i \in [r]} |\alpha_i|^2 \cdot \lambda_i \geq c(|x|) . \end{aligned}$$

If $|\phi\rangle \in W_x^\perp$ is a pure state then by similar arguments we get $\|\Pi_{acc}V_x(|\phi\rangle \otimes |0^m\rangle)\|^2 \leq w(|x|)$.

When $x \notin L$, condition 2 of Definition 2 gets satisfied analogously from condition 2 of Definition 4.

Part 2 (Definition 2 \Rightarrow Definition 4): Let $L \in \text{FewQMA}(c, w, s)$ with some verification procedure $\{V_x\}$ and polynomial q according to Definition 2. We claim that $\{V_x\}$ and q satisfy also Definition 4 with the same parameters.

First consider the case $x \in L$. The cardinality of the dimension of the subspace of witnesses W_x in $\mathcal{B}^{\otimes k}$ is in $[q(|x|)]$ by hypothesis. We set

$$W_c = \text{span}\{|v\rangle \in \mathcal{B}^{\otimes k} : |v\rangle \otimes |0^m\rangle \text{ is an eigenvector of } \Pi_x \text{ with eigenvalue } \geq c(|x|)\} ,$$

and

$$W_w = \text{span}\{|v\rangle \in \mathcal{B}^{\otimes k} : |v\rangle \otimes |0^m\rangle \text{ is an eigenvector of } \Pi_x \text{ with eigenvalue } > w(|x|)\} .$$

We will show that $\dim(W_x) = \dim(W_c)$ and that $W_c = W_w$, from which the claim follows. For this, it is sufficient to prove that $\dim(W_c) = \dim(W_x) = \dim(W_w)$, since clearly W_c is a subspace of W_w .

First observe that the definitions of W_c and W_w imply that

$$W_c^\perp = \text{span}\{|v\rangle \in \mathcal{B}^{\otimes k} : |v\rangle \otimes |0^m\rangle \text{ is an eigenvector of } \Pi_x \text{ with eigenvalue } < c(|x|)\} ,$$

and

$$W_w^\perp = \text{span}\{|v\rangle \in \mathcal{B}^{\otimes k} : |v\rangle \otimes |0^m\rangle \text{ is an eigenvector of } \Pi_x \text{ with eigenvalue } \leq w(|x|)\} .$$

Let us suppose that $\dim(W_x) < \dim(W_c)$. Then there exists a vector $|u\rangle$ in $W_c \cap W_x^\perp$. Since $|u\rangle \in W_c$, using arguments as in Part 1 above, we have $\|\Pi_{acc}V_x(|u\rangle \otimes |0^m\rangle)\|^2 \geq c(|x|)$. However,

since $|u\rangle \in W_x^\perp$, from condition 1(b) of Definition 2 we have $\|\Pi_{acc}V_x(|u\rangle \otimes |0^m\rangle)\|^2 \leq w(|x|) < c(|x|)$ which is a contradiction. We similarly reach a contradiction assuming $\dim(W_x) > \dim(W_c)$ and hence $\dim(W_x) = \dim(W_c)$.

The equality $\dim(W_w) = \dim(W_x)$ can be proven by an argument analogous to the proof of $\dim(W_x) = \dim(W_c)$.

In the case $x \notin L$, assume for contradiction that there is an eigenvalue $\lambda > s(|x|)$ of Π_x with eigenvector $|v\rangle \otimes |0^m\rangle$. Then as before,

$$\begin{aligned}\|\Pi_{acc}V_x(|\psi\rangle \otimes |0^m\rangle)\|^2 &= \|\Pi_{acc}V_x\Pi_{init}(|v\rangle \otimes |0^m\rangle)\|^2 \\ &= (\langle v| \otimes \langle 0^m|)\Pi_x(|v\rangle \otimes |0^m\rangle) = \lambda > s(|x|) ,\end{aligned}$$

which contradicts condition 2 of Definition 2. \square

The alternative definition of $\text{FewQMA}(c, w, s)$ is useful in arriving at the following strong error probability reduction theorem whose proof follows very similar lines as the QMA strong error probability reduction proof in Marriott and Watrous [23] and hence is skipped.

Theorem 2 (Error reduction) *Let $c, w, s : \mathbb{N} \rightarrow [0, 1]$ be polynomial-time computable functions such that for some polynomial p , for all n , they satisfy $c(n) > \max\{w(n), s(n)\} + 1/p(n)$. Let r be any polynomial. Then for any $L \in \text{FewQMA}(c, w, s)$ having a verification procedure with polynomials k, m, q , there exists a verification procedure for L with parameters $(c', w', s') = (1 - 2^{-r}, 2^{-r}, 2^{-r})$ and polynomials $k' = k$, $m' = \text{poly}(m, r)$ and $q' = q$.*

Definition 5 *A promise problem $L = (L_{yes}, L_{no})$ is in the complexity class Unique Quantum Merlin-Arthur UQMA if L is in FewQMA with the additional constraint that for all $x \in L_{yes}$, the subspace W_x of witnesses in the definition of FewQMA is one-dimensional.*

Definition 6 *UQMA-CPP is the promise problem $(\text{UQMA-CPP}_{yes}, \text{UQMA-CPP}_{no})$ where the elements of $\text{UQMA-CPP}_{yes} \cup \text{UQMA-CPP}_{no}$ are descriptions of quantum circuits V with k witness qubits and m auxiliary qubits, such that*

1. $V \in \text{UQMA-CPP}_{yes}$ if
 - (a) there exists a witness $|\psi\rangle$ such that $\Pr[V \text{ outputs Accept on } |\psi\rangle] \geq 2/3$,
 - (b) for all witnesses $|\phi\rangle$ orthogonal to $|\psi\rangle$, $\Pr[V \text{ outputs Accept on } |\phi\rangle] \leq 1/3$,
2. $V \in \text{UQMA-CPP}_{no}$ if for all pure states $|\psi\rangle$, $\Pr[V \text{ outputs Accept on } |\psi\rangle] \leq 1/3$.

It is easily verified that UQMA-CPP is the canonical UQMA-complete promise problem.

4 Reducing the dimension of the witness subspace

This section will be entirely devoted to the proof of our main result.

Theorem 3 (Main Theorem) $\text{FewQMA} \subseteq \text{P}^{\text{UQMA}}$.

Proof

Let $L \in \text{FewQMA}$ have a verification procedure $\{V'_x : x \in \{0, 1\}^*\}$ with polynomials k, m', q . Let r be a polynomial such that $q2^{-r} \leq 1/3$. Then, we know from Theorem 2 that $L \in \text{FewQMA}(1 - 2^{-r}, 2^{-r}, 2^{-r})$ with verification procedure $\{V_x : x \in \{0, 1\}^*\}$ and polynomials k, m, q .

Our goal is to describe a deterministic polynomial-time algorithm \mathcal{A} , with access to the oracle \mathcal{O} for the promise problem UQMA-CPP, that decides the promise problem L . In high level, our algorithm works in the following way. On input x and for all $t \in [q(|x|)]$, \mathcal{A} calls \mathcal{O} with a quantum circuit A_x^t that uses $t \cdot k$ witness qubits and $t \cdot m$ auxiliary qubits, and outputs **Accept** if and only if the witness has the following two properties: first, it belongs to the alternating subspace of $\mathcal{H}^{\otimes t}$ and second the circuit V_x , when performed on each of the t registers separately, outputs **Accept** on all of them. \mathcal{A} accepts iff for any t , oracle \mathcal{O} accepts. We will prove that for $x \in L_{yes}$, we have $A_x^d \in \text{UQMA-CPP}_{yes}$, where $d = \dim(W_x)$. Hence \mathcal{O} accepts A_x^t and therefore \mathcal{A} accepts. On the other hand, for $x \in L_{no}$, we show that for all $t \in [q(|x|)]$, $A_x^t \in \text{UQMA-CPP}_{no}$ and hence \mathcal{A} rejects.

We first describe in detail the **Alternating Test** and the **Witness Test** that appear in the algorithm. In our descriptions below k, m, q, r represent the integers $k(|x|), m(|x|), q(|x|)$ and $r(|x|)$ respectively.

4.1 Alternating Test

Let \mathcal{H} be the Hilbert space $\mathcal{B}^{\otimes k}$ and let $t \in [2^k]$. Let us fix some polynomial-time computable bijection between the set $[t!]$ and the set of permutations S_t . Let \mathcal{P}_t be the $(t!)$ -dimensional Hilbert space spanned by vectors $|i\rangle$, for $i \in [t!]$. We will use the elements of S_t for describing the above basis vectors via the fixed bijection.

The **Alternating Test** with parameter t receives, as input, a pure state in $\mathcal{H}^{\otimes t}$ and performs a unitary operation in the Hilbert space $\mathcal{P}_t \otimes \mathcal{H}^{\otimes t}$, followed by a measurement. We will refer to the elements of $\mathcal{P}_t \otimes \mathcal{H}^{\otimes t}$ as consisting of two registers R and S , where the content of each register is a mixed state with support over the corresponding Hilbert space.

Let us define $|\text{perm}_t\rangle = \frac{1}{\sqrt{t!}} \sum_{\pi \in S_t} \text{sgn}(\pi) |\pi\rangle$.

Alternating Test(t)

Input: A pure state $|\psi\rangle \in \mathcal{H}^{\otimes t}$ in the $(t \cdot k)$ -qubit register S
Output: The content of register S and **Accept** or **Reject**.

1. Create the state $(\frac{1}{\sqrt{t!}} \sum_{\pi \in S_t} |\pi\rangle) \otimes |\psi\rangle$.
2. Apply the unitary $U : |\pi\rangle \otimes |\psi\rangle \rightarrow |\pi\rangle \otimes U_{\pi}|\psi\rangle$.
3. Perform the measurement $(M, I - M)$, where $M = |\text{perm}_t\rangle \langle \text{perm}_t| \otimes I_{\mathcal{H}^{\otimes t}}$. Output the content of S . Output **Accept** if the state has been projected onto M and output **Reject** otherwise.

It is easily verified that the **Alternating Test(t)** runs in time polynomial in $t \cdot k$. Since we will only call it with $t \in [q]$, its running time will be polynomial in $|x|$. The following lemma states that the **Alternating Test(t)** is a projection onto the subspace $\text{Alt}^{\mathcal{H}^{\otimes t}}$.

Lemma 1 *1. For any pure state $|\psi\rangle \in \text{Alt}^{\mathcal{H}^{\otimes t}}$, the Alternating Test(t) outputs the state $|\psi\rangle$ and **Accept** with probability 1.*
*2. For any $|\phi\rangle \in (\text{Alt}^{\mathcal{H}^{\otimes t}})^{\perp}$, the Alternating Test(t) outputs **Accept** with probability 0.*

Proof Part 1: Since $|\psi\rangle \in \text{Alt}^{\mathcal{H}^{\otimes t}}$ we have $U_{\pi}|\psi\rangle = \text{sgn}(\pi) \cdot |\psi\rangle$, and therefore the state after Step 2 is

$$\frac{1}{\sqrt{t!}} \sum_{\pi \in S_t} |\pi\rangle \otimes U_{\pi}|\psi\rangle = \frac{1}{\sqrt{t!}} \sum_{\pi \in S_t} \text{sgn}(\pi) \cdot |\pi\rangle \otimes |\psi\rangle = |\text{perm}_t\rangle \otimes |\psi\rangle.$$

Part 2: By Claim 1, we have $(\text{Alt}^{\mathcal{H}^{\otimes t}})^\perp = \sum_{i \neq j} \text{Sym}_{ij}^{\mathcal{H}^{\otimes t}}$. Hence it is enough to show that for all distinct $i, j \in [t]$ and for any vector $|\phi\rangle \in \text{Sym}_{ij}^{\mathcal{H}^{\otimes t}}$, the probability p that the Alternating Test(t) outputs **Accept** is 0. We have

$$\begin{aligned} p &= \left(\frac{1}{\sqrt{t!}} \sum_{\sigma \in S_t} \langle \sigma | \otimes \langle \phi | U_\sigma^\dagger\right) \left(|\text{perm}_t\rangle \langle \text{perm}_t| \otimes I_{\mathcal{H}^{\otimes t}}\right) \left(\frac{1}{\sqrt{t!}} \sum_{\pi \in S_t} |\pi\rangle \otimes U_\pi |\phi\rangle\right) \\ &= \left(\frac{1}{t!}\right)^2 \sum_{\sigma, \pi \in S_t} \text{sgn}(\sigma) \cdot \text{sgn}(\pi) \cdot \langle \phi | U_\sigma^\dagger U_\pi | \phi \rangle \end{aligned}$$

We define $\pi' = \pi \circ \pi_{ij}$. Then $\pi = \pi' \circ \pi_{ij}^{-1} = \pi' \circ \pi_{ij}$ and $\text{sgn}(\pi) = -\text{sgn}(\pi')$. Since $|\phi\rangle \in \text{Sym}_{ij}^{\mathcal{H}^{\otimes t}}$, we have $U_{\pi_{ij}} |\phi\rangle = |\phi\rangle$ and hence $U_{\pi' \circ \pi_{ij}} |\phi\rangle = U_{\pi'} \cdot (U_{\pi_{ij}} |\phi\rangle) = U_{\pi'} |\phi\rangle$. Therefore,

$$\begin{aligned} p &= \left(\frac{1}{t!}\right)^2 \sum_{\sigma, \pi \in S_t} \text{sgn}(\sigma) \cdot \text{sgn}(\pi) \cdot \langle \phi | U_\sigma^\dagger U_\pi | \phi \rangle = (-1) \cdot \left(\frac{1}{t!}\right)^2 \sum_{\sigma, \pi' \in S_t} \text{sgn}(\sigma) \cdot \text{sgn}(\pi') \cdot \langle \phi | U_\sigma^\dagger U_{\pi' \circ \pi_{ij}} | \phi \rangle \\ &= (-1) \cdot \left(\frac{1}{t!}\right)^2 \sum_{\sigma, \pi' \in S_t} \text{sgn}(\sigma) \cdot \text{sgn}(\pi') \cdot \langle \phi | U_\sigma^\dagger U_{\pi'} | \phi \rangle = -p \end{aligned}$$

Hence $p = 0$. \square

4.2 Witness Test

The Witness Test with parameter $t \in [q]$ receives as input a pure state in $\mathcal{H}^{\otimes t}$ and performs a unitary operation in the Hilbert space $\mathcal{H}^{\otimes t} \otimes \mathcal{B}^{\otimes (tm)}$ followed by a measurement. We will refer to the elements of $\mathcal{H}^{\otimes t} \otimes \mathcal{B}^{\otimes (tm)}$ as consisting of t pairs of registers (T_i, Z_i) respectively on k and m qubits, for $i \in [t]$. All registers Z_i will be initialized to $|0^m\rangle$.

Witness Test(t)

Input: A pure state $|\psi\rangle \in \mathcal{H}^{\otimes t}$ in the k -qubit registers T_i , for $i \in [t]$
Output: **Accept** or **Reject**

1. For all $i \in [t]$, append a register Z_i initialized to $|0^m\rangle$ and apply the circuit V_x on registers (T_i, Z_i) .
2. Output **Accept** if for all $i \in [t]$, V_x outputs **Accept**; otherwise output **Reject**.

We can describe the Witness Test(t) as the operator $(\Pi_{acc} V_x)^{\otimes t}$ acting on a state $|\psi\rangle \otimes |0^{tm}\rangle$. Hence, $\Pr[\text{Witness Test}(t) \text{ outputs } \text{Accept} \text{ on } |\psi\rangle] = \|(\Pi_{acc} V_x)^{\otimes t}(|\psi\rangle \otimes |0^{tm}\rangle)\|^2$. Note that the description of the circuit $V_x^{\otimes t}$ can be generated in polynomial-time, since the circuit family $\{V_x, x \in \{0, 1\}^*\}$ is uniformly generated in polynomial-time.

In what follows we will have to argue about the probability that the verification procedure V_x outputs **Accept** when its input is some mixed state. Even though we have only considered pure states as inputs in the definition of the class **FewQMA**, we will see that it is not hard to extend our arguments to mixed states.

Lemma 2 1. If $x \in L_{yes}$, then for every $|\psi\rangle \in W_x^{\otimes t}$, the Witness Test(t) outputs **Accept** with probability at least $2/3$.

2. If $x \in L_{yes}$, then for every $|\phi\rangle \in (W_x^{\otimes t})^\perp$, the Witness Test(t) outputs Accept with probability at most $1/3$.
3. If $x \in L_{no}$, then for every $|\psi\rangle \in \mathcal{H}^{\otimes t}$, the Witness Test(t) outputs Accept with probability at most $1/3$.

Proof Part 1: By completeness we know that for any pure state $|\psi'\rangle \in W_x$, we have

$\Pr[V_x \text{ outputs Reject on } |\psi'\rangle] \leq 2^{-r}$. Let ρ_i denote the reduced density matrix of $|\psi\rangle$ on register T_i . Since $|\psi\rangle \in W_x^{\otimes t}$, then for every $i \in [t]$, the density matrix ρ_i is a distribution of pure states that all belong to W_x and hence $\Pr[V_x \text{ outputs Reject on } \rho_i] \leq 2^{-r}$. It follows from the union bound that

$$\Pr[\text{Witness Test}(t) \text{ outputs Accept on } |\psi\rangle] \geq 1 - \sum_{i=1}^t \Pr[V_x \text{ outputs Reject on } \rho_i] \geq 1 - t \cdot 2^{-r} \geq 2/3,$$

where the last inequality follows from the choice of r .

Part 2: For $i \in [t]$, let $S_i = W_x \otimes \dots \otimes W_x \otimes W_x^\perp \otimes \mathcal{H} \dots \otimes \mathcal{H}$, where W_x^\perp stands in the i^{th} component of the tensor product. By Fact 1 we have $(W_x^{\otimes t})^\perp = \bigoplus_{i \in [t]} S_i$.

Let us therefore consider a pure state $|\phi\rangle \in \bigoplus_{i \in [t]} S_i$ and let $|\phi\rangle = \sum_{i \in [t]} a_i |\phi_i\rangle$, where $|\phi_i\rangle$ is a pure state in S_i . Then $\sum_{i=1}^t |a_i|^2 = 1$ because the $|\phi_i\rangle$'s are also orthogonal. Furthermore, let ρ_i be the reduced density matrix of $|\phi_i\rangle$ on register T_i . Then the support of ρ_i is over W_x^\perp . Since for any pure state $|\phi'\rangle \in W_x^\perp$ we have $\Pr[V_x \text{ outputs Accept on } |\phi'\rangle] \leq 2^{-r}$, we conclude that $\Pr[V_x \text{ outputs Accept on } \rho_i] \leq 2^{-r}$.

The probability that the Witness Test(t) outputs Accept on $|\phi_i\rangle$ is equal to the probability that all t applications of V_x output Accept, which is less than the probability that the i^{th} application of V_x outputs Accept, since the projections, performed in different registers, commute. Hence,

$$\Pr[\text{Witness Test}(t) \text{ outputs Accept on } |\phi_i\rangle] \leq \Pr[V_x \text{ outputs Accept on } \rho_i] \leq 2^{-r}.$$

Now for the input $|\phi\rangle$ we have

$$\begin{aligned} \Pr[\text{Witness Test}(t) \text{ outputs Accept on } |\phi\rangle] &= \|(\Pi_{acc} V_x)^{\otimes t} (|\phi\rangle \otimes |0^{tm}\rangle)\|^2 \\ &= \left\| \sum_{i \in [t]} a_i (\Pi_{acc} V_x)^{\otimes t} (|\phi_i\rangle \otimes |0^{tm}\rangle) \right\|^2 \leq \left(\sum_{i \in [t]} |a_i| \cdot \|(\Pi_{acc} V_x)^{\otimes t} (|\phi_i\rangle \otimes |0^{tm}\rangle)\| \right)^2 \\ &\leq \left(\sum_{i \in [t]} |a_i|^2 \right) \cdot \left(\sum_{i \in [t]} \|(\Pi_{acc} V_x)^{\otimes t} (|\phi_i\rangle \otimes |0^{tm}\rangle)\|^2 \right) \leq t \cdot 2^{-r} \leq 1/3. \end{aligned}$$

In the above calculation the first two inequalities follow respectively from the triangle inequality and the Cauchy-Schwarz inequality.

Part 3: By the soundness of the original protocol we know that for any pure state $|\psi'\rangle \in \mathcal{H}$ $\Pr[V_x \text{ outputs Accept on } |\psi'\rangle] \leq 2^{-r}$. The same holds for any mixed state as well. Since the probability that the Witness Test(t) outputs Accept is at most the probability that the procedure V_x accepts the state on the first register T_1 we conclude that

$$\Pr[\text{Witness Test}(t) \text{ outputs Accept on } |\psi\rangle] \leq \Pr[V_x \text{ outputs Accept on } \rho_1] \leq 2^{-r} \leq 1/3. \quad \square$$

4.3 Putting it all together

Finally we describe the algorithm \mathcal{A} in the figure below and proceed to analyze its properties.

Algorithm \mathcal{A}

Input: $x \in L_{yes} \cup L_{no}$
 Output: Accept or Reject

1. For $t = 1, \dots, q(|x|)$ do:
 - (a) Call the oracle \mathcal{O} with input A_x^t , where A_x^t is the description of the circuit of the following procedure on $t \cdot k$ witness qubits and $t \cdot m$ auxiliary qubits

Input: A pure state $|\psi\rangle \in \mathcal{H}^{\otimes t}$; Output: Accept or Reject

 - i. Run the Alternating Test(t) with input $|\psi\rangle$.
 - ii. Run the Witness Test(t) with input being the output state of the Alternating Test(t).
 - iii. Output Accept iff both Tests output Accept.
 - (b) If \mathcal{O} outputs Accept then output Accept and halt.
2. Output Reject.

Running time: We have seen that the description of the circuit that performs the Alternating Test(t) can be generated in time polynomial in $t \cdot k$ which is polynomial in $|x|$. The description of the circuit that performs the Witness Test(t) can also be generated in polynomial-time, since the circuit family $\{V_x : x \in \{0,1\}^*\}$ can be generated uniformly in polynomial-time. Hence the description of the circuit A_x^t can be generated in polynomial-time and the overall algorithm \mathcal{A} runs in polynomial-time.

Correctness in case $x \in L_{yes}$: Let us consider the oracle call with input A_x^d where d is the dimension of W_x . We prove that $A_x^d \in \text{UQMA-CPP}_{yes}$, hence the oracle \mathcal{O} outputs Accept and therefore \mathcal{A} outputs Accept as well. Our claim is immediate from the following lemma.

Lemma 3 1. $\Pr[A_x^d \text{ outputs Accept on } |W_{alt}\rangle] \geq 2/3$.

2. Let $|\phi\rangle \in \mathcal{H}^{\otimes d}$ be orthogonal to $|W_{alt}\rangle$. Then $\Pr[A_x^d \text{ outputs Accept on } |\phi\rangle] \leq 1/3$.

Proof Part (1): Since $|W_{alt}\rangle \in \text{Alt}^{\mathcal{H}^{\otimes d}}$, Lemma 1 tells us that the Alternating Test(d) outputs the state $|W_{alt}\rangle$ and Accept with probability 1. Then, since for every $i \in [d]$ the support of the reduced density matrix of $|W_{alt}\rangle$ on register T_i is on W_x , Lemma 2 tells us that the Witness Test outputs Accept with probability at least $2/3$.

Part (2): By Claim 4 and the fact that the state $|\phi\rangle$ is orthogonal to $|W_{alt}\rangle$, we can conclude that if the Alternating Test(d) outputs Accept then the output state is a pure state $|\phi'\rangle \in (\mathcal{H}^{\otimes d})^\perp$. Now, by Lemma 2, the probability that the Witness Test(d) outputs Accept on input $|\phi'\rangle$ is at most $1/3$. \square

Correctness in case $x \in L_{no}$: By Lemma 2 it follows easily that for all $t \in [q] : A_x^t \in \text{UQMA-CPP}_{no}$. In this case, \mathcal{O} outputs Reject in every iteration, and hence \mathcal{A} outputs Reject.

This concludes the proof of Theorem 3. \square

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A Yet another definition of FewQMA

We have seen two possible definitions for the class **FewQMA** and have proven their equivalence. In high level, one says that in the yes instances there is a polynomial number of eigenvalues of the projection operator Π_x that are larger than $2/3$, while no eigenvalue is in the interval $(1/3, 2/3)$. The other definition says that in a yes instance there exists a subspace of polynomial dimension such that every state in the subspace is accepted with probability at least $2/3$ and every state orthogonal to this subspace is accepted with probability at most $1/3$.

A natural question is whether the use of a subspace is necessary in the second definition or we could have just talked about a set of orthonormal vectors of polynomial size, where each vector is accepted with probability $2/3$ and every vector orthogonal to these ones is accepted with probability at most $1/3$. While we are unable to show the equivalence of the complexity class defined this way and **FewQMA**, a weak equivalence can indeed be shown. For this we include the parameters of the verification procedure and the bound on the number of witnesses in the definition of the class, and we also require strong amplification. More precisely, consider the following definition.

Definition 7 Let $c, w, s : \mathbb{N} \rightarrow [0, 1]$ be polynomial time computable functions such that $c(n) > \max\{w(n), s(n)\}$ for all $n \in \mathbb{N}$. Let q, k, m be polynomials such that $q(n) \leq 2^{k(n)}$ for all $n \in \mathbb{N}$. A promise problem $L = (L_{\text{yes}}, L_{\text{no}})$ is in the complexity class Vector Few Quantum Merlin-Arthur Vector-FewQMA(c, w, s, q, k, m) if there exists a verification procedure $\{V_x : x \in \{0, 1\}^*\}$ with k witness qubits and m ancilla qubits, such that

1. for all $x \in L_{\text{yes}}$ there exists an orthonormal basis $\{|\psi_1\rangle, \dots, |\psi_{2^k}\rangle\}$ of the witness space and $d \in [q(|x|)]$ such that
 - (a) for all pure states $|\psi_i\rangle$ with $i \in [d]$, $\|\Pi_{\text{acc}}V_x(|\psi_i\rangle \otimes |0^m\rangle)\|^2 \geq c(|x|)$,
 - (b) for all pure states $|\psi_i\rangle$ with $d+1 \leq i \leq 2^k$, $\|\Pi_{\text{acc}}V_x(|\psi_i\rangle \otimes |0^m\rangle)\|^2 \leq w(|x|)$,
2. for all $x \in L_{\text{no}}$ and for all pure states $|\psi\rangle \in \mathcal{B}^{\otimes k}$, $\|\Pi_{\text{acc}}V_x(|\psi\rangle \otimes |0^m\rangle)\|^2 \leq s(|x|)$.

Finally we define

$$\text{Vector-FewQMA} = \bigcup_{\substack{\text{polynomials } q, k, m: \\ q \leq 2^k}} \text{Vector-FewQMA}\left(1 - \frac{1}{3q}, \frac{1}{3 \cdot 2^k}, \frac{1}{3}, q, k, m\right) .$$

We show **Vector-FewQMA** = **FewQMA**. For showing the equivalence we use Horn's Theorem that states that for a Hermitian matrix, the vector of the eigenvalues majorizes the diagonal.

Theorem 4 [13] Let R be a natural number. Let $\Lambda = \{\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_R | \lambda_i \in \mathbb{R}\}$ and $A = \{\mu_1 \geq \mu_2 \geq \dots \geq \mu_R | \mu_i \in \mathbb{R}\}$. Then there exists an $R \times R$ Hermitian matrix with set of eigenvalues Λ and set of diagonal elements A if and only if $\sum_{i=1}^t (\lambda_i - \mu_i) \geq 0$ for all $t \in [R]$ and with equality for $t = R$.

Theorem 5

$$\text{FewQMA} = \text{Vector-FewQMA} .$$

Proof We first show **FewQMA** \subseteq **Vector-FewQMA**. Let $L \in \text{FewQMA}$ have a verification procedure $\{V'_x : x \in \{0, 1\}^*\}$ with polynomials k, m', q . Let r be a polynomial such that $2^{-r} \leq \frac{1}{3 \cdot 2^k}$ and $2^{-r} \leq \frac{1}{3q}$. We know from Theorem 2 that $L \in \text{FewQMA}(1 - 2^{-r}, 2^{-r}, 2^{-r})$ with verification procedure $\{V_x : x \in \{0, 1\}^*\}$ and polynomials k, m, q , where $m = \text{poly}(m', r)$. Then the eigenbasis of Π_x satisfies the requirements of the definition of **Vector-FewQMA** $(1 - \frac{1}{3q}, \frac{1}{3 \cdot 2^k}, \frac{1}{3}, q, k, m)$. This shows that $L \in \text{Vector-FewQMA}$.

We now show **Vector-FewQMA** \subseteq **FewQMA**. Let $L \in \text{Vector-FewQMA}(1 - \frac{1}{3q}, \frac{1}{3 \cdot 2^k}, \frac{1}{3}, q, k, m)$, for some polynomials q, k, m such that $q \leq 2^k$. Let $N = 2^k$ and $\mathcal{H} = \mathcal{B}^{\otimes k}$. If $x \in L_{\text{yes}}$, then there exist an orthonormal basis $\{|\psi_1\rangle, \dots, |\psi_N\rangle\}$ for \mathcal{H} and $d \in [q]$, such that for $i \in [d]$, $\mu_i \geq 1 - \frac{1}{3q}$ and for $d+1 \leq i \leq N$, $\mu_i \leq \frac{1}{3N}$, where by definition

$$\mu_i \stackrel{\text{def}}{=} \|\Pi_{\text{acc}}V_x(|\psi_i\rangle \otimes |0^m\rangle)\|^2 = \langle \psi_i | \otimes \langle 0^m | \Pi_x | \psi_i \rangle \otimes |0^m\rangle .$$

Consider now the Hermitian matrix M that describes the projection operator Π_x in a basis that is an extension of $\{|\psi_1\rangle \otimes |0^m\rangle, \dots, |\psi_N\rangle \otimes |0^m\rangle\}$. Note that $\mu_i, i \in [N]$ are the first N diagonal elements of M . Observe that an eigenvector of Π_x with non-zero eigenvalue is also an eigenvector of Π_{init} with non-zero eigenvalue. Since there are N non-zero eigenvalues of Π_{init} , there are at most N non-zero eigenvalues of Π_x . This also implies that $\mu_i = 0$ for $N < i \leq 2^{k+m}$. Let the first N eigenvalues of Π_x in decreasing order be $\lambda_i, i \in [N]$. Then, using Horn's theorem,

$$\sum_{i=1}^d \lambda_i \geq \sum_{i=1}^d \mu_i \geq d \cdot \left(1 - \frac{1}{3q}\right)$$

which implies (since $\lambda_i \leq 1$, for all $i \in [N]$) that

$$\lambda_d \geq -(d-1) + d \cdot \left(1 - \frac{1}{3q}\right) \geq 1 - \frac{d}{3q} \geq \frac{2}{3} .$$

Also, we have that

$$\lambda_{d+1} \leq \sum_{i=d+1}^{2^{k+m}} \lambda_i \leq \sum_{i=d+1}^{2^{k+m}} \mu_i = \sum_{i=d+1}^{2^k} \mu_i \leq (N-d) \frac{1}{3N} \leq \frac{1}{3} .$$

If $x \in L_{no}$ then by the soundness condition $\lambda_1 \leq \frac{1}{3}$. This shows that $L \in \text{FewQMA}$. \square